Equivalent Off–Diagonal Cosmological Models and Ekpyrotic Scenarios in f(R)–Modified, Massive and Einstein Gravity

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Abstract

We reinvestigate how generic off–diagonal cosmological solutions depending, in general, on all spacetime coordinates can be constructed in massive and f–modified gravity using the anholonomic frame deformation method. There are constructed new classes of locally anisotropic and (in) homogeneous cosmological metrics with open and closed spatial geometries. By resorting such solutions, we show that they describe the late time acceleration due to effective cosmological terms induced by nonlinear off–diagonal interactions, possible modifications of the gravitational action and graviton mass. The cosmological metrics and related Stückelberg fields are constructed in explicit form up to nonholonomic frame transforms of the Friedmann–Lamaître–Robertson–Walker (FLRW) coordinates. The solutions include matter, graviton mass and other effective sources modelling nonlinear gravitational and matter fields interactions with polarization of physical constants and deformations of metrics, which may explain dark energy and dark matter effects. However, we argue that it is not obligatory always to modify gravity if we consider effective generalized Einstein equations with nontrivial vacuum and/or non-minimal coupling with matter. Indeed, we state certain conditions when such configurations mimic interesting solutions in general relativity and modifications, for instance, when we can extract the general Painlevé–Gullstrand and FLRW metrics. In a more general context, we elaborate on a reconstruction procedure for off–diagonal cosmological solutions which describe cyclic and ekpyrotic universes. Finally, there are discussed open issues and further perspectives.

Keywords: modified gravity, massive gravity, off–diagonal cosmological solutions; ekpyrotic and little rip universe.

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Contents

1 Introduction

The "ekpyrotic" and "new ekpyrotic" mechanisms and cyclic models have been elaborated as alternatives to standard big bank inflationary cosmology [2]. Another alternative came from the idea that graviton may have a nontrivial mass as it was proposed in the Fierz and Pauli work [1]. For recent reviews and related f(R) modifications and cosmological models, in general, with non-minimal coupling and dilaton-brane cosmology, local anisotropies and/or effective modelling of massive gravity, see respectively [3], [4] and [5]. The modern version of a ghost free (bimetric) massive gravity theory were made in a series of papers (recent reviews can be found in [6]): Developing generic nolinear versions of the Fierz-Pauli theory, the so–called vDVZ discontinuity problem was solved via the Vainshtein mechanism [7] (avoiding discontinuity by going beyond the linear theory). There are also more recent approaches based on DGP model [8]. During a long time, none solution was found for another problem with ghosts because at nonlinear order in massive gravity appears a sixth scalar degree of freedom as a ghost. This problem was cosndiered in a paper by Boulware and Deser [9] together with similar issues related to the effective field theory approach.

A considerable amount of work has been made in order to undestand the implications and find possible applications of of such ghost–free models. The most substantial progress was made when de Rham and co–authors had shown how to eliminate the scalar mode and Hassan and Rosen provided a complete proof for a class of bigravity / bimetric gravity theories, see [10]. In such approaches, the second metric describes an effective exotic matter related to massive gravitons and does not suffer from ghost instability to all orders in a perturbation theory and away from the decoupling limit.

It is important to mention that the first black hole solutions in a nonlinear massive theory were found in the context of high–energy physics and for off–diagonal and/or higher dimensions generalizations [11]. A general decoupling of generalized Einstein equations can be proven following the anholonomic frame deformation methods, AFDM, [12]. Here we also note that the possibility that the graviton has a nonzero mass $\mathring{\mu}$ results not only in fundamental theoretical implications but give rise to straightforward phenomenological and cosmological consequences. For instance, a gravitational potential of Yukawa form $\sim e^{-\mathring{\mu}r}/r$ results in decay of gravitational interactions at scales $r \geq \mathring{\mu}^{-1}$ and such an effect may result in the accelerated expansion of the Universe. This way, a massive gravity theory, MGT, provides alternatives to dark energy and, via effective polarizations of fundamental physical constants may explain certain dark matter effects. We can treat this as a result of generic off–diagonal nonlinear interactions. Recently, various cosmological models derived

for ghost free (modified) massive gravity and bigravity theories have been elaborated and studied intensively [13, 3, 14].

The goal of this work is to construct generic off–diagonal cosmological solutions in MGT and state the conditions when such configurations are modelled equivalently in general relativity (GR). It extends the letter variant of [15] with new results and more details on proofs of results and constructing exact solutions and further analysis of results and speculation on reconstruction mechanism.

As a first step, we consider off–diagonal deformations of a "prime" cosmological solution taken in general Painlevé–Gullstrand (PG) form, when the Friedman–Lamaître–Robertson–Walker (FLRW) can be recast for well–defined geometric conditions. Such constructions are performed in section 2.

At the second step, the "target" metrics will be generated to possess one Killing symmetry (or other none Killing symmetries) and depend on timelike and certain (all) spacelike coordinates. In general, such off–diagonal solutions are with local anisotropy and inhomogeneities for effective cosmological constants and polarizations of other physical constants and coefficients of cosmological metrics which can be modelled both in MGT and GR. We consider the method of constructing generic off–diagonal cosmological solutions in section 3.

Then (the third step), we shall emphasize and speculate on importance of off–diagonal nonlinear gravitational interactions for elaborating cosmological scenarios when dark matter and dark energy effects can be explained by anisotropic polarizations of vacuum and/or de Sitter like configurations. We provide examples of off–diagonal solutions with solitonic configurations, see section 4.

A reconstruction mechanism for off–diagonal cosmological solutions with modified gravity and/or massive graviton effects is elaborated in section 5. Finally, we conclude the paper in section 6.

2 Equivalent modelling of f–modified and massive gravity theories

We shall work with MGTs modelled on a pseudo–Riemannian spacetime $\mathbf{V}$ with physical metric $\mathbf{g} = \{\mathbf{g}_{\mu\nu}\}$ and fiducial metrics. On massive gravity, see reviews [6] and on geometric methods in gravity and constructing exact solutions, see Refs. [12] . In addition to well–known approaches with diadic and tetradic (vierbein) variables, we shall work with nonholonomic manifolds when certain classes of frame transforms can be adapted to preserve a chosen splitting of the nonlinear and linear connection structures into some standard components (for instance, defining the Levi Civita, LC, connection) and distortion tensors which can be fixed to be zero if additional constraints are imposed.

Using different nonholonomic frame variables, the action for our model can be written in two forms,

$$S = \frac{1}{16\pi}\int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|}[\widehat{f}(\widehat{R}) - \frac{\mathring{\mu}^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu},\mathbf{K}_{\alpha\beta}) + {}^mL] \quad (1)$$

$$= \frac{1}{16\pi}\int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|}[f(R) + {}^mL]. \quad (2)$$

In above formulas, the physical/geometrical objects with "hat" and/or written in "boldface" form are considered for a conventional $2+2$ splitting when the two dimensional horizontal, h, and the two dimensional vertical, v, coordinates are labelled (respectively) in the form $u^\alpha = (x^i, y^a)$, or $u = (x, y)$, with indices $i, j, k... = 1, 2$ and $a, b, ... = 3, 4$. The scalar curvature R is for the LC–connection ∇.

We write by $\widehat{R}$ the scalar curvature for an auxiliary (canonical) connection $\widehat{\mathbf{D}}$ uniquely determined by two conditions:

1. it is metric compatible, $\widehat{\mathbf{D}}\mathbf{g} = 0$, and

2. the h- and v-torsions are zero (but there are nonzero $h-v$ components of torsion $\widehat{\mathcal{T}}$ completely determined by $\mathbf{g}$) for a conventional splitting

$$\mathbf{N}: \ T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V},$$

with local coefficients of the so–called nonlinear connection, N–connection structure, labeled in the form $\mathbf{N} = \{N_i^b\}$.

General frame transforms can be parameterized in the form $e_\alpha = A_{\ \alpha}^{\alpha'}(u)\partial_{\alpha'}$, where the matix $A_{\ \alpha}^{\alpha'}$ is non–degenerate in a finite, or infinite region of $\mathbf{V}$ and $\partial_{\alpha'} = \partial/\partial u^{\alpha'}$. Using such $A_{\ \alpha}^{\alpha'}$, we can always re–define the geometric and physical object with respect to a class of N–adapted (dual) bases

$$\begin{aligned} \mathbf{e}_\alpha &= (\mathbf{e}_i = \partial_i - N_i^b\partial_b, e_a = \partial_a = \partial/\partial y^a) \text{ and} \\ \mathbf{e}^\beta &= (e^j = dx^i, \mathbf{e}^b = dy^b + N_c^b dy^c), \end{aligned} \tag{3}$$

which are nonholonomic (equivalently, anholonomic) because, in general, there are satisfied relations of type

$$\mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma,$$

for certain nontrivial anholonomy coefficients $W_{\alpha\beta}^\gamma(u)$. The Einstein summation rule on repeating indices will be applied if the contrary is not stated.

The connection $\widehat{\mathbf{D}}$ allows us to decouple the field equations in various gravity theories and construct exact solutions in very general forms. Here we note that the distortion relation from the LC connection

$$\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}[\widehat{\mathcal{T}}]$$

is uniquely determined by a distorting tensor $\widehat{\mathbf{Z}}$ completely defined by $\widehat{\mathcal{T}}$ and (as a consequence for such models) by $(\mathbf{g}, \mathbf{N})$. The main idea of the AFDM [12] is to use $\widehat{\mathbf{D}}$ as an auxiliary [for the (pseudo) Riemennian spacetimes] and/or [with nonholonomically induced torsion] connection which positively allows us to decouple gravitational field equations for very general conditions with respect to N–adapted frames (3). We can not perform such a decoupling in general form if we work from the very beginning with the data $(\nabla, \partial_{\alpha'})$ but there are proofs that this is possible for $(\widehat{\mathbf{D}}, \mathbf{e}_\alpha; \mathbf{g}, \mathbf{N})$. Having constructed integral varieties (for instance, locally anisotropic and/or inhomogeneous cosmological ones), we can impose additional nonholonomic (non–integrable constraints) when $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \rightarrow \nabla$ and $\widehat{R} \rightarrow R$, where R is the scalar curvature of ∇, and it is possible to extract exact solutions in GR.[1]

The MGTs with actions of type (1) generalize the so–called modified $f(R)$ gravity and the ghost–free massive gravity [10]. We shall follow some conventions from [14]. There will be used the units when $\hbar = c = 1$ and the Planck mass M_{Pl} is defined via $M_{Pl}^2 = 1/8\pi G$, with 4–d Newton constant G. We write δu^4 instead of d^4u because there are used N–elongated differentials as in (3). A model can be specified by corresponding constants, dynamical physical equations and their solutions in corresponding variables. It will be considered that $\mathring{\mu} = const$ is the mass of graviton. For LC–configurations, we can fix conditions of type

$$\widehat{f}(\widehat{R}) - \frac{\mathring{\mu}^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu}, \mathbf{K}_{\alpha\beta}) = f(\widehat{R}), \quad \text{or} \quad \widehat{f}(\widehat{R}) = f(R), \quad \text{or} \quad \widehat{f}(\widehat{R}) = R, \tag{4}$$

which depend on the type of models we elaborate and what classes of solutions we wont to construct. Such conditions can be very general ones for arbitrary frame transforms and N–connection deformations. We emphasize that it is possible to find solutions in explicit form if we chose the coefficients

[1] There will be considered also left and right up/low indices as labels for some geometric/physical objects.

$\{N_i^a\}$ and the local frames for $\widehat{\mathbf{D}}$ when $\widehat{R} = const$ in such forms that $\partial_\alpha \widehat{f}(\widehat{R}) = (\partial_{\widehat{R}} \widehat{f}) \times \partial_\alpha \widehat{R} = 0$ but, in general, $\partial_\alpha f(R) \neq 0$.

The equations of motion for our nonholonomically modified massive gravity theory can be written

$$(\partial_{\widehat{R}} \widehat{f}) \widehat{\mathbf{R}}_{\mu\nu} - \frac{1}{2} \widehat{f}(\widehat{R}) \mathbf{g}_{\mu\nu} + \mathring{\mu}^2 \mathbf{X}_{\mu\nu} = M_{Pl}^{-2} \mathbf{T}_{\mu\nu}, \tag{5}$$

where M_{Pl} is the Plank mass, $\widehat{\mathbf{R}}_{\mu\nu}$ is the Einstein tensor for a pseudo–Riemannian metric $\mathbf{g}_{\mu\nu}$ and $\widehat{\mathbf{D}}$, $\mathbf{T}_{\mu\nu}$ is the standard matter energy–momentum tensor. We note that in above formulas, for $\widehat{\mathbf{D}} \to \nabla$, we get $\widehat{\mathbf{R}}_{\mu\nu} \to R_{\mu\nu}$ with a standard Ricci tensor $R_{\mu\nu}$ for ∇. Such limits give rise to original modified massive gravity models but it is important to find some generalized solutons and only at the end to consider additional assumptions on limits and nonholonomic constraints. It should be emphasized that the effective energy–momentum tensor $\mathbf{X}_{\mu\nu}$ in (5) is defined by the potential of graviton $\mathcal{U} = \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4$, where α_3 and α_4 are free parameters. The values $\mathcal{U}_2, \mathcal{U}_3$ and $\mathcal{U}_4$ are certain polynomials on traces of some other polynomials of a matrix $\mathcal{K}_\mu^\nu = \delta_\mu^\nu - \left(\sqrt{g^{-1}\Sigma}\right)_\mu^\nu$ for a tensor determined by four Stückelberg fields $\phi^{\underline{\mu}}$ as

$$\Sigma_{\mu\nu} = \partial_\mu \phi^{\underline{\mu}} \partial_\nu \phi^{\underline{\nu}} \eta_{\underline{\mu\nu}}, \tag{6}$$

when $\eta_{\underline{\mu\nu}} = (1, 1, 1, -1)$. A series of arguments presented in [14] (geometrically, we cand consider corresponding nonholonomic frame transforms and nonholonomic variables) prove that the parameter choice $\alpha_3 = (\alpha - 1)/3, \alpha_4 = (\alpha^2 - \alpha + 1)/12$ is useful for avoiding potential ghost instabilities. By frame transforms, we can fix

$$\mathbf{X}_{\mu\nu} = \alpha^{-1} \mathbf{g}_{\mu\nu}. \tag{7}$$

By explicit computations, we can prove that for configurations (7), de Sitter solutions with effective cosmological constant are possible, for instance, for ansatz of PG type,

$$ds^2 = U^2(r,t)[dr + \epsilon \sqrt{f(r,t)} dt]^2 + \widetilde{\alpha}^2 r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - V^2(r,t) dt^2. \tag{8}$$

In above formula, there are used spherical coordinates labelled in the form $u^\beta = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$, when the function f takes non–negative values and the constant $\widetilde{\alpha} = \alpha/(\alpha+1)$ and $\epsilon = \pm 1$. We can consider bimetric configurations (determined as soltuons of the system 5), 6) and (7)) with Stückelberg fields parameterized in the unitary gauge as $\phi^{\underline{4}} = t$ and $\phi^{\underline{1}} = r\widehat{n}^{\underline{1}}, \phi^{\underline{2}} = r\widehat{n}^{\underline{2}}, \phi^{\underline{3}} = r\widehat{n}^{\underline{3}}$, where a three dimensional (3–d) unit vector is defined as $\widehat{n} = (\widehat{n}^{\underline{1}} = \sin\theta\cos\varphi, \widehat{n}^{\underline{2}} = \sin\theta\sin\varphi, \widehat{n}^{\underline{3}} = \cos\theta)$.

First we note that any PG metric of type (8) defines solutions both in GR and in MGT. For instance, we can extract the de Sitter solution, in the absence of matter, and obtain standard cosmological equations with FLRW metric, for a perfect fluid source

$$T_{\mu\nu} = [\rho(t) + p(t)]\, u_\mu u_\nu + p(t) g_{\mu\nu}, \tag{9}$$

where $u_\mu = (0, 0, 0, -V)$ can be reproduced for the effective cosmological constant ${}^{eff}\lambda = \mathring{\mu}^2/\alpha$. Secondly, it is also possible to express metrics of type (8) in a familiar cosmological FLRW form (see formulas (23), (24) and (27) in [14]).

Finally, in this section, we note that off–diagonal deformations of such solutions can be constructed for different classes of zero graviton mass and/or massive theories and various f–modifications, see examples in [12] and [11].

3 Generating off–diagonal cosmological solutions

We now make a crucial assumption that our Universe can be described by inhomogeneous cosmological metrics are with Killing symmetry on $\partial_3 = \partial_\varphi$ and, in general, can not be diagonalized by coordinate transforms. As a matter of principle, we can consider dependencies on all spacetime coordinates but this require more cumbersome computations, see examples in Refs. [12]. Inhomogeneities and anisotropies can be very small, but it is important to find certain classes of general solutions for generic nonlinear systems and impose at the end certain homogeneity and high symmetry conditions by selecting corresponding subclasses of generation and integration functions. Up to general classes of frame transforms, we can consider the ansatz

$$\begin{aligned} ds^2 &= \eta_1(r,\theta)\mathring{g}_1(r)dr^2 + \eta_2(r,\theta)\mathring{g}_2(r)d\theta^2 + \omega^2(r,\theta,\varphi,t)\{\eta_3(r,\theta,t) \times \\ & \mathring{h}_3(r,\theta)[d\varphi + n_i(r,\theta)dx^i]^2 + \eta_4(r,\theta,t)\mathring{h}_4(r,\theta,t)[dt + (w_i(x^k,t) + \mathring{w}_i(x^k))dx^i]^2\}. \end{aligned} \quad (10)$$

The values η_α are called "polarization" functions, where ω is the so–called "vertical", v, conformal factor.

For metrics (10), we can consider off–diagonal N–coefficients labelled $N_i^a(x^k, y^4)$, where (for parameterizations corresponding to this class of ansatz)

$$N_i^3 = n_i(r,\theta) \text{ and } N_i^4 = w_i(x^k,t) + \mathring{w}_i(x^k).$$

The data for the "primary" metric are

$$\begin{aligned} \mathring{g}_1(r) &= U^2 - \mathring{h}_4(\mathring{w}_1)^2, \mathring{g}_2(r) = \widetilde{\alpha}^2 r^2, \mathring{h}_3 = \widetilde{\alpha}^2 r^2 \sin^2\theta, \mathring{h}_4 = \sqrt{|fU^2 - V^2|}, \\ \mathring{w}_1 &= \epsilon\sqrt{f}U^2/\mathring{h}_4, \mathring{w}_2 = 0, \mathring{n}_i = 0, \end{aligned} \quad (11)$$

when the coordinate system is such way fixed that the values f, U, V in (8) result in a coefficient $\mathring{g}_1$ depending only on r.

Using nonholonomic frame transforms, we can parameterize the energy momentum sources (9) and effective (7) in the form

$$\Upsilon^\alpha_\beta = \frac{1}{M^2_{Pl}(\partial_{\widehat{R}}\widehat{f})}(\mathbf{T}^\alpha_\beta + \alpha^{-1}\mathbf{X}^\alpha_\beta) = \frac{1}{M^2_{Pl}(\partial_{\widehat{R}}\widehat{f})}(\ ^mT + \alpha^{-1})\delta^\alpha_\beta = (\ ^m\Upsilon + \ ^\alpha\Upsilon)\delta^\alpha_\beta, \quad (12)$$

for constant values $^m\Upsilon := M^{-2}_{Pl}(\partial_{\widehat{R}}\widehat{f})^{-1}\ ^mT$ and $^\alpha\Upsilon = M^{-2}_{Pl}(\partial_{\widehat{R}}\widehat{f})^{-1}\alpha^{-1}$, with respect to N–adapted frames (3). In general, such sources are not diagonal and may depend on all spacetime coordinates. Our assumption is that we prescribe a distribution with one Killing symmetry in a moment of time and then find further evolution with respect to certain classes of nonholonomic frames. We emphasize that fixing such N–adapted parameterizations we can decouple the gravitational filed equations in MGTs and construct exact solutions in explicit form.

Let us outline in brief the decoupling property of the gravitational and matter field equations in GR and various generalizations/ modifications studied in details in Refs. [12]. That anholonomic frame deformation method (AFDM) can be applied for decoupling, and constructing solutions of the MGT field equations (5) with any effective source parameterized in the form (12), see details in Refs. [12]).

We label the target off–diagonal metrics as $\mathbf{g} = (g_i = \eta_i\mathring{g}_i, h_a = \eta_a\mathring{h}_a, N^a_j)$ with coefficients determined by ansatz (10). In these formulas, there is not summation on repeating indices in this formula. We shall use brief denotations for partial derivatives: $\partial_1\psi = \psi^\bullet, \partial_2\psi = \psi', \partial_3\psi = \psi^\diamond$ and

$\partial_4\psi = \psi^*$. Computing the N–adapted coefficients of the Ricci tensors, when $\widehat{R}^1_1 = \widehat{R}^2_2, \widehat{R}^3_3 = \widehat{R}^4_4, \widehat{R}_{3k}$ and $\widehat{R}_{4k}$ are not trivial, we write (5) as a system of nonlinear partial differential equations (PDE):

$$\begin{aligned} \psi^{\bullet\bullet} + \psi'' &= 2(\ ^m\Upsilon + \ ^\alpha\Upsilon), \quad (13)\\ \phi^* h_3^* &= 2h_3h_4(\ ^m\Upsilon + \ ^\alpha\Upsilon),\\ n_i^{**} + \gamma n_i^* &= 0, \beta w_i - \alpha_i = 0,\\ \partial_k\omega &= n_k\omega^\diamond + w_k\omega^*, \quad (14)\end{aligned}$$

for

$$\phi = \ln\left|\frac{h_3^*}{\sqrt{|h_3h_4|}}\right|, \gamma := (\ln\frac{|h_3|^{3/2}}{|h_4|})^*, \quad \alpha_i = \frac{h_3^*}{2h_3}\partial_i\phi,\ \beta = \frac{h_3^*}{2h_3}\phi^*, \quad (15)$$

In above formulas, we consider the system of coordinates and polarization functions are fixed for configurations with $g_1 = g_2 = e^{\psi(x^k)}$ and nonzero values ϕ^* and h_a^*.

We can extract solutions for the LC–configurations with zero torsion if the coefficients of metrics are subjected to additional conditions:

$$w_i^* = \mathbf{e}_i \ln\sqrt{|\ h_4|}, \mathbf{e}_i \ln\sqrt{|\ h_3|} = 0, \partial_i w_j = \partial_j w_i \text{ and } n_i^* = 0. \quad (16)$$

Step by step, the system of nonlinear PDE (13)–(16) can be integrated in general forms for any ω constrained by a system of linear first order equations (14), see details in [12]. The explicit solutions are given by quadratic elements

$$ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \frac{\Phi^2\omega^2}{4\ (\ ^m\Upsilon + \ ^\alpha\Upsilon)}\mathring{h}_3[d\varphi + (\partial_k\ n)\,dx^k]^2 - \frac{(\Phi^*)^2\omega^2}{(\ ^m\Upsilon + \ ^\alpha\Upsilon)\Phi^2}\mathring{h}_4[dt + (\partial_i\ \widetilde{A})dx^i]^2. \quad (17)$$

for any

$$\Phi = \check{\Phi}, (\partial_i\check{\Phi})^* = \partial_i\check{\Phi}^*, w_i + \mathring{w}_i = \partial_i\check{\Phi}/\check{\Phi}^* = \partial_i\ \widetilde{A}.$$

We can construct exact solutions even such conditions are not satisfied, i.e. the zero torsion conditions are not stated or there are given certain sources in non–explicit form. So, the AFDM can be applied to generate both off–diagonal metrics and nonholonomically induced torsions. There are various physical arguments for what type of generating/ integration functions and sources we have to chose in order to construct realistic scenarios for Universe acceleration and observable dark energy/ matter effects.

Coming back to the properties of general solutions, we note that we can generate new classes of solutions for arbitrary nontrivial sources, $\ ^m\Upsilon + \ ^\alpha\Upsilon \neq 0$, and generating functions, $\Phi(x^k, t) := e^\phi$ and $n_k = \partial_k n(x^i)$. Resulting target metrics are generic off–diagonal and can not be diagonalized via coordinate transforms in a finite spacetime region because, in general, the anholonomy coefficients $W^\gamma_{\alpha\beta}$ for (3) are not zero (we can check by explicit computations). The polarization η–functions from (17) are

$$\eta_1 = e^\psi/\mathring{g}_1, \eta_2 = e^\psi/\mathring{g}_2, \eta_3 = \Phi^2/4\ (\ ^m\Upsilon + \ ^\alpha\Upsilon), \eta_4 = (\Phi^*)^2/(\ ^m\Upsilon + \ ^\alpha\Upsilon)\Phi^2. \quad (18)$$

We conclude that prescribing any generating functions $\check{\Phi}(r,\theta,t)$, $n(r,\theta)$, $\omega(r,\theta,\varphi,t)$ and sources $^m\Upsilon$, $^\alpha\Upsilon$ and then computing $\check{A}(r,\theta,t)$, we can transform any PG (and, similarly, FLRW) metric $\mathring{\mathbf{g}} = (\mathring{g}_i, \mathring{h}_a, \mathring{w}_i, \mathring{n}_i)$ in MGT and/or GR into new classes of generic off–diagonal exact solutions depending on all spacetime coordinates. Such metrics define Einstein manifolds in GR with effective cosmological constants determined by $\ ^m\Upsilon + \ ^\alpha\Upsilon$. With respect to N–adapted frames (3) the coefficients of metric encode contributions from massive gravity, determined by $\ ^\alpha\Upsilon$, and matter fields, included in $\ ^m\Upsilon$.

We also note that it is possible to provide an "alternative" treatment of (17) as exact solutions in MGT. In such a case, we have to define and analyze the properties of fiducial Stückelberg fields $\phi^{\underline{\mu}}$ and the corresponding bimetric structure resulting in target solutions $\mathbf{g} = (g_i, h_a, N_j^a)$:

Let us analyze the primary configurations related to

$$\mathring{\phi}^{\underline{\mu}} = (\mathring{\phi}^{\underline{i}} = a(\tau)\rho\widetilde{\alpha}^{-1}\widehat{n}^{\underline{i}}, \mathring{\phi}^{\underline{3}} = a(\tau)\rho\widetilde{\alpha}^{-1}\widehat{n}^{\underline{3}}, \mathring{\phi}^{\underline{4}} = \tau\kappa^{-1}),$$

when the corresponding prime PG–metric $\mathring{\mathbf{g}}$ is taken in FLRW form

$$ds^2 = a^2(d\rho^2/(1-K\rho^2) + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)) - d\tau^2.$$

A corresponding fiducial tensor (6) is computed

$$\mathring{\Sigma}_{\underline{\mu\nu}} du^{\underline{\mu}} du^{\underline{\nu}} = \frac{a^2}{\widetilde{\alpha}^2}[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2) + 2H\rho d\rho d\tau - (\frac{\widetilde{\alpha}^2}{\kappa^2 a^2} - H^2\rho^2)d\tau^2],$$

where the coefficients and coordinates are re–defined in the form $r \to \rho = \widetilde{\alpha}r/a(\tau)$ and $t \to \tau = \kappa t$, for $K = 0, \pm 1; \kappa$ is an integration constant; $H := d\ln a/d\tau$ and the local coordinates are parameterized in the form $x^{\underline{1}} = \rho, x^{\underline{2}} = \theta, y^{\underline{3}} = \varphi, y^{\underline{4}} = \tau$.

At the next step, for a target metric $\mathbf{g} = \mathbf{g}_{\alpha\beta}$ and frames $\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\ \underline{\alpha}}\partial_{\underline{\alpha}}$, we write

$$\mathbf{g}_{\alpha\beta} = \mathbf{e}_\alpha^{\ \underline{\alpha}}\mathbf{e}_\beta^{\ \underline{\beta}}\eta_{\underline{\alpha\beta}} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b g_{ab} & N_i^a h_{ab} \\ N_i^a h_{ab} & h_{ab} \end{array}\right], \text{ for } \mathbf{e}_\alpha^{\ \underline{\alpha}} = \left[\begin{array}{cc} \mathbf{e}_i^{\ \underline{i}} & N_i^b \mathbf{e}_b^{\ \underline{a}} \\ 0 & \mathbf{e}_a^{\ \underline{a}} \end{array}\right].$$

The values

$$g_{ij} = \mathbf{e}_i^{\ \underline{\alpha}}\mathbf{e}_j^{\ \underline{\beta}}\eta_{\underline{\alpha\beta}} = e^{\psi}\delta_{ij} = diag[\eta_i \mathring{g}_i], h_{ab} = \mathbf{e}_i^{\ \underline{\alpha}}\mathbf{e}_j^{\ \underline{\beta}}\eta_{\underline{\alpha\beta}} = diag[\eta_a \mathring{h}_a], N_i^3 = \partial_k\ n, N_i^4 = \partial_k\ \widetilde{A}$$

are related algebraically to data (18) resulting in off–diagonal solutions (17). Then, to work out the the "target" Stückelberg fields we compute $\phi^{\mu'} = \mathbf{e}^{\mu'}_{\ \ \underline{\mu}}\ \phi^{\underline{\mu}}$ with $\mathbf{e}^{\mu'}_{\ \ \underline{\mu}}$ being inverse to $\mathbf{e}_\alpha^{\ \underline{\alpha}}$, and the fiducial tensor

$$\Sigma_{\alpha\beta} = (\mathbf{e}_\alpha\phi^{\underline{\mu}})(\mathbf{e}_\beta\phi^{\underline{\nu}})\eta_{\underline{\mu\nu}} = \mathbf{e}_\alpha^{\ \underline{\alpha}}\mathbf{e}_\beta^{\ \underline{\beta}}\Sigma_{\underline{\alpha\beta}}.$$

If the prime value $\mathring{\Sigma}_{\underline{\mu\nu}}$ carries information about two constants κ and $\widetilde{\alpha}$, a target tensor $\Sigma_{\mu\nu}$ is associated to off–diagonal solutions and encodes data about generating and integration functions and via superpositions on possible Killing symmetries, on various integration constants. Similar constructions were elaborated for holonomic and nonholonomic configurations in GR, see [16].

In the framework of MGT, two cosmological solutions $\mathring{\mathbf{g}}$ and $\mathbf{g}$ related by nonholonomic deformations[2] are characterised respectively by two invariants

$$\mathring{I}^{\underline{\alpha\beta}} = \mathring{g}^{\alpha\beta}\partial_\alpha\mathring{\phi}^{\underline{\alpha}}\partial_\beta\mathring{\phi}^{\underline{\beta}} \text{ and } \mathbf{I}^{\alpha\beta} = \mathbf{g}^{\alpha\beta}\mathbf{e}_\alpha\phi^{\underline{\alpha}}\mathbf{e}_\beta\phi^{\underline{\beta}}.$$

The tensor $\mathring{I}^{\underline{\alpha\beta}}$ does not contain singularities because there are not coordinate singularities on horizon for PG metrics. It should be emphasized that the symmetry of $\Sigma_{\mu\nu}$ is not the same as that of $\mathring{\Sigma}_{\underline{\mu\nu}}$ and the singular behaviour of $\mathbf{I}^{\alpha\beta}$ depends on the class of generating and integration functions we chose of constructing a target solution $\mathbf{g}$.

In GR, MGTs and/or Einstein–Finsler gravity theories [12], off–cosmological solutions of type (17) were found to generalized various models of Biachi, Kasner, Gödel and other universes. There are known locally anisotropic black hole and wormhole, in general, with solitonic background solutions, see [11, 17]. For instance, Bianchi type anisotropic cosmological metrics are generated if we impose corresponding Lie algebra symmetries on metrics. It was emphasized in [14] that "any PG–type solution in general relativity (with a cosmological constant) is also a solution to massive gravity." Such

[2]involving not only frame transforms but also deformation of the linear connection structure when at the end there are imposed additional constraints for zero torsion

a conclusion can be extended to a large class of generic off–diagonal cosmological solutions generated by effective cosmological constants but it is not true, for instance, if we consider nonholonomic deformations with nonholonomically induced torsion like in metric compatible Finsler theories.

Finally, we note that the analysis of cosmological perturbations around an off–diagonal cosmological background is not trivial because the fiducial and reference metrics do not respect the same symmetries. Nevertheless, fluctuations around de Sitter backgrounds seem to have a decoupling limit which implies that one can avoid potential ghost instabilities if the parameter choice is considered both for diagonal and off–diagonal cosmological solutions, see details in [18]. This special choice also allows us to have a structure $X_{\mu\nu} \sim g_{\mu\nu}$ at list in N–adapted frames when the massive gravity effects can be approximated by effective cosmological constants and exact solutions in MGT which are also solutions in GR.

4 Examples of off–diagonal solutions with solitonic configurations

We now consider three examples of off–diagonal cosmological solutions with solitonic modifications in MGT and (with alternative interpretation) GR. Two and three dimensional solitonic waves are typical nonlinear wave configurations which can be used for generating spacetime metrics with Killing, or non–Killing, symmetries and can be characterised by additional parametric dependencies and solitonic symmetries. Moving solitonic configurations can mimic various types of modified gravity dark energy and dark matter effects with nontrivial gravitational vacuum, polarization of constants and additional nonlinear diagonal and off–diagonal interactions of the gravitational and matter fields.

4.1 One soliton solutions

We shall for simplicity work with a nonlinear radial (solitonic, with left s-label) generating function

$$\Phi = {}^{s}\check{\Phi}(r,t) = 4\arctan e^{q\sigma(r-vt)+q_0} \tag{19}$$

and $\omega = 1$, we construct a metric

$$\mathbf{ds}^2 = e^{\psi(r,\theta)}(dr^2 + d\theta^2) + \frac{{}^{s}\check{\Phi}^2}{4\,(\,{}^{m}\Upsilon + {}^{\alpha}\Upsilon)}\mathring{h}_3(r,\theta)d\varphi^2 - \frac{(\partial_t\, {}^{s}\check{\Phi})^2}{(\,{}^{m}\Upsilon + {}^{\alpha}\Upsilon)\, {}^{s}\check{\Phi}^2}\mathring{h}_4(r,t)[dt + (\partial_r\, \widetilde{A})dr]^2, \tag{20}$$

In this metric, for simplicity, we fixed $n(r,\theta) = 0$ and consider that $\widetilde{A}(r,t)$ is defined as a solution of ${}^{s}\check{\Phi}^{\bullet}/\,{}^{s}\check{\Phi}^{*} = \partial_r\, \widetilde{A}$ and $\mathring{h}_a$ are given by PG–data (11). The generating function (19), where $\sigma^2 = (1-v^2)^{-1}$ for constants q, q_0, v, is just a 1–soliton solution of the sine–Gordon equation

$${}^{s}\check{\Phi}^{**} - {}^{s}\check{\Phi}^{\bullet\bullet} + \sin\, {}^{s}\check{\Phi} = 0.$$

For any class of small polarizations with $\eta_a \sim 1$), we can consider that the source $(\,{}^{m}\Upsilon + {}^{\alpha}\Upsilon)$ is polarized by ${}^{s}\check{\Phi}^{-2}$ when $h_3 \sim \mathring{h}_3$ and $h_4 \sim \mathring{h}_4(\,{}^{s}\check{\Phi}^{*})^2/\,{}^{s}\check{\Phi}^{-4}$ with an off–diagonal term $\partial_r\, \widetilde{A}$ resulting in a stationary solitonic universe. If we consider that $(\partial_{\widehat{R}} \widehat{f})^{-1} = {}^{s}\check{\Phi}^{-2}$ in (12), we can model $\widehat{f}$–interactions of type (1) via off–diagonal interactions and "gravitational polarizations".

In absence of matter, ${}^{m}\Upsilon = 0$, the off–diagonal cosmology is completely determined by ${}^{\alpha}\Upsilon$ when ${}^{s}\check{\Phi}$ transforms $\mathring{\mu}$ into an anisotropically polarized/variable mass of solitonic waves. Such configurations can be modelled if ${}^{m}\Upsilon \ll {}^{\alpha}\Upsilon$. If ${}^{m}\Upsilon \gg {}^{\alpha}\Upsilon$, we generate cosmological models determined by distribution off matter fields when contributions from massive gravity are with small

anisotropic polarization. For a class of nonholonomic constraints on Φ and ψ (which may be not of solitonic type), when solutions (20) are of type (10) with $\eta_\alpha \sim 1$ and $n_i, w_i \sim 0$, we approximate PG–metrics of type (8).

Hence, by an appropriate choice of generating functions and sources, we can model equivalently modified gravity effects, massive gravity contributions or matter field configurations in GR and MGT interactions. For well defined conditions, such configurations can be studied in the framework of some classes of off–diagonal solutions in Einstein gravity with effective cosmological constants.

4.2 Three dimensional solitonic anistoropic waves

More sophisticate nonlinear gravitational and matter filed interactions can be modeled both in MGTs and GR if we consider more general classes of solutions of effective Einstein equations.

For instance, off–diagonal solitonic metrics with one Killing symmetry can be generated, for instance, if we take instead of (19) a generating functions ${}^s\check{\Phi}(r,\theta,t)$ which is a solution of the Kadomtsev–Petivashvili, KdP, equations [19],

$$\pm\ {}^s\check{\Phi}'' + (\ {}^s\check{\Phi}^* +\ {}^s\check{\Phi}\ \ {}^s\check{\Phi}^\bullet + \epsilon\ {}^s\check{\Phi}^{\bullet\bullet\bullet})^\bullet = 0,$$

when solutions induce certain anisotropy on θ.[3] In the dispersionless limit $\epsilon \to 0$, we can consider that the solutions are independent on θ and determined by Burgers' equation

$${}^s\check{\Phi}^* +\ {}^s\check{\Phi}\ \ {}^s\check{\Phi}^\bullet = 0.$$

Such solutions can be parameterized and treated similarly to (20) but with, in general, a nontrivial term $(\partial_\theta\ \widetilde{A})d\theta$ after $\mathring{h}_4$, when ${}^s\check{\Phi}^\bullet/\ {}^s\check{\Phi}^* =\ \widetilde{A}^\bullet$ and ${}^s\check{\Phi}'/\ {}^s\check{\Phi}^* =\ \widetilde{A}'$. Similar metrics were constructed for Dirac spinor waves and solitons in anisotropic Taub-NUT spaces and in five dimensional brane gravity which can be encoded and classified by corresponding solitonic hierarchies and geometric invariants, see [17]. Here we proved that AFDM can extended to generate inhomogeneous off–diagonal cosmological solitonic solutions in varios MGTs.

4.3 Solitonic waves for a nontrivial vertical conformal v–factor

The cosmological solutions we look are also with three dimensional solitons but for a v–factor as in (10). For instance, we consider solitons of KdP type, when $\omega = \check{\omega}(r,\varphi,t)$, when $x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t$, for

$$\pm\ \check{\omega}^{\diamond\diamond} + (\partial_t \check{\omega} + \check{\omega}\ \check{\omega}^\bullet + \epsilon \check{\omega}^{\bullet\bullet\bullet})^\bullet = 0, \qquad (21)$$

In the dispersionless limit $\epsilon \to 0$, we can consider that the solutions are independent on angle φ and determined by Burgers' equation

$$\check{\omega}^* + \check{\omega}\ \check{\omega}^\bullet = 0.$$

The conditions (14) impose an additional constraint

$$\mathbf{e}_1 \check{\omega} = \check{\omega}^\bullet + w_1(r,\theta,\varphi)\check{\omega}^* + n_1(r,\theta)\check{\omega}^\diamond = 0.$$

In the system of coordinates when $\check{\omega}' = 0$, we can fix $w_2 = 0$ and $n_2 = 0$. For any arbitrary generating function with LC–configuration, $\check{\Phi}(r,\theta,t)$, we construct exact solutions

$$\mathbf{ds}^2 = e^{\psi(r,\theta)}(dr^2 +\ d\theta^2) + \frac{\check{\Phi}^2 \check{\omega}^2}{4\ (\ {}^m\Upsilon +\ {}^\alpha\Upsilon)} \mathring{h}_3(r,\theta) d\varphi^2 - \frac{(\partial_t\ \check{\Phi})^2 \check{\omega}^2}{(\ {}^m\Upsilon +\ {}^\alpha\Upsilon)\ \check{\Phi}^2} \mathring{h}_4(r,t)[dt + (\partial_r\ \widetilde{A})dr]^2, \qquad (22)$$

[3] In a similar form, we can construct various types of vacuum gravitational 2-d and 3-d configurations characterized by solitonic hierarchies and related bi–Hamilton structures, for instance, of KdP equations with possible mixtures with solutions for 2-d and 3-d sine–Gordon equations etc, see details in Ref. [17].

which are generic off–diagonal and depend on all spacetime coordinates. Such stationary cosmological solutions are with polarizations on two angles θ and φ. Nevertheless, the character of anisotropies is different for metrics of type (20) and (22). In the third class of metrics, we obtain a Killing symmetry on ∂_φ only in the limit $\breve{\omega} \to 1$, but in the first two ones, such a symmetry exists generically. For (22), the value $\check{\Phi}$ is not obligatory a solitonic one which can be used for additional off–diagonal modifications of solutions and various types of polarizations.

We can provide a physical interpretation of 22 which is similar to 20 if the generating and integration functions are chosen to satisfy the conditions $\eta_\alpha \sim 1$ and $n_i, w_i \sim 0$, we approximate PG–metrics of type (8).

In a particular case, we can use a conformal v–factor which is a 1–solitonic one, i.e.

$$\breve{\omega} \to \omega(r,t) = 4 \arctan e^{q\sigma(r-vt)+q_0},$$

where $\sigma^2 = (1-v^2)^{-1}$ and constants q, q_0, v, defines a 1–soliton solution of the sine–Gordon equation

$$\omega^{**} - \omega^{\bullet\bullet} + \sin\omega = 0.$$

Such a soliton propagates in time along the radial coordinate.

We conclude that solitonic waves may mimic both particle type configurations as dark matter and encode certain hidden dark energy and off–diagonal gavitational and matter field interactions.

5 Reconstruction mechanism for off–cosmological solutions

We now consider a reconstruction mechanism with distinguished off–diagonal cosmological effects [12] by generalizing some methods elaborated for $f(R)$ gravity in [3]. The main idea was to present the MGT actions (with zero or non-zero gravitational mass) as sums of actions in GR and certain effective ideal fluid contributions with parameters defined by nontrivial $\mathring{\mu}$ and f–deformations. The reconstruction method was developed for such formulations which lead to cosmology with cyclic evolution. Then, it was proven that the ekpyrotic scenario may be also realized for MGTs and that it is possible to reconstruct models of $f(R)$ gravity which induces little rip cosmology.

In our approach, we work with cosmological generic off–diagonal metrics and generalized connections. We can always chose such generating functions and parameters of solutions that off–diagonal contributions are small, and the torsion is constrained to be zero, beginning a certain fixed moment of time. Nevertheless, we can not neglect for such classes of solutions possible nonlinear effects determined by generating functions and effective sources resulting in diagonalized modifications. MGTs were elaborated as realistic alternatives for unified description of inflation with dark energy when cosmological scenarios encode information on various massive and zero–mass gravitational modes. A crucial question is if and how such constructions have to be elaborated for generic off–diagonal cosmological spaces. This is not only a geometric problem for generalizing the reconstruction formalism for inhomogeneous and locally anisotropic cosmological theories. It is connected to a very important question on equivalent modeling of cosmological scenarios for different MGTs in the framework of GR with nonholonomic and off–diagonal nonlinear interactions.

Any cosmological solution in massive, MGT and/or GR parameterized in a form (10) (in particular, as (20) and (22)) can be encoded into an effective functional (4) when

$$\widehat{f} - \frac{\mathring{\mu}^2}{4}\mathcal{U} = f(\widehat{R}), \widehat{R}_{|\widehat{\mathbf{D}}\to\nabla} = R.$$

This allows us to work as in MGT; the conditions $\partial_\alpha f(\widehat{R}) = 0$ if $\widehat{R} = const$ simplify substantially the computations. Nevertheless, the contributions of the parameter $\mathring{\mu}$ and effective potential $\mathcal{U}$ are

included as functional dependencies of f and $\widehat{R}$.For $\widehat{\mathbf{D}} \to \nabla$, we get nonlinear modifications both in diagonal and off–diagonal terms of cosmological metrics.

The starting point of our approach is to consider a prime flat FLRW like metric

$$ds^2 = a^2(t)[(dx^1)^2 + (dx^2)^2 + (dy^3)^2] - dt^2,$$

where t is the cosmological time. In order to extract a monotonically expanding and periodic cosmological scenario, we parameterize $\ln |a(t)| = H_0 t + \tilde{a}(t)$ for a periodic function $\tilde{a}(t+\tau) = {}^1a \cos(2\pi t/\tau)$, where $0 < {}^1a < H_0$. Our goal is to prove that such a behavior is encoded into off–diagonal solutions of type (20)–(22).

We write FLRW like equations with respect to N–adapted (moving) frames (3) for a generalized Hubble function H,

$$3H^2 = 8\pi\rho \text{ and } 3H^2 + 2\mathbf{e}_4 H = -8\pi p.$$

Using variables with $\partial_\alpha f(\widehat{R})_{|\widehat{R}=const} = 0$, we can consider a function $H(t)$ when $\mathbf{e}_4 H = \partial_t H = H^*$. It should be noted that the approximation $\mathbf{e}_4 \to \partial_t$ is considered "at the end" (after a class of off-diagonal solutions was found for a necessary type connection, which allowed to decouple the equations) and the generating functions and effective sources are constrained to depend only on t. The energy–density and pressure of an effective perfect fluid are computed

$$\begin{aligned} \rho &= (8\pi)^{-1}[(\partial_R f)^{-1}(\frac{1}{2}f(R) + 3H\mathbf{e}_4(\partial_R f)) - 3\mathbf{e}_4 H] \qquad (23)\\ &= (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\widehat{f}|} - 3H^*] = (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R})|} - 3H^*], \\ p &= -(8\pi)^{-1}[(\partial_R f)^{-1}(\frac{1}{2}f(R) + 2H\mathbf{e}_4(\partial_R f) + \mathbf{e}_4\mathbf{e}_4(\partial_R f)) + \mathbf{e}_4 H] \\ &= (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\widehat{f}|} + H^*] = -(8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R})|} + H^*]. \end{aligned}$$

We emphasize that, in general, such values are defined with respect to nonholonomic frames. The effect of nonlinear deformations encoding physical data $(\mathring{\mu}, \mathcal{U}, f, \widehat{R})$ is preserved also in diagonal contributions for $\mathbf{e}_4 \to \partial_t$.

The equation of state, EoS, parameter for the effective dark fluid encoding MGTs parameters is defined by

$$w = \frac{p}{\rho} = \frac{\widehat{f} + 2H^*\partial_{\widehat{R}}\widehat{f}}{\widehat{f} - 6H^*\partial_{\widehat{R}}\widehat{f}} = \frac{\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R}) + 2H^*\partial_{\widehat{R}}\widehat{f}}{\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R}) - 6H^*\partial_{\widehat{R}}\widehat{f}}, \qquad (24)$$

when the corresponding EoS is

$$p = -\rho - (2\pi)^{-1} H^*$$

and $\mathcal{U}(t)$ is computed, for simplicity, for a configuration of "target" Stückelberg fields $\phi^{\mu'} = \mathbf{e}^{\mu'}{}_{\underline{\mu}}\, \phi^{\underline{\mu}}$ when a found solution is finally modelled by generating functions with dependencies on t. The components of $\phi^{\mu'}$ are computed with respect to N–adapted frames.

Taking a generating Hubble parameter $H(t) = H_0 t + H_1 \sin \omega t$, for $\omega = 2\pi/\tau$, we can recover the modified action for oscillations of off–diagonal (massive) universe (see similar details in [3]),

$$f(R(t)) = 6\omega H_1 \int dt[\omega \sin \omega t - 4 \cos \omega t(H_0 + H_1 \sin \omega t)] \exp[H_0 t + \frac{H_1}{\omega} \sin \omega t]. \qquad (25)$$

Here we note that we can not invert analytically to find in explicit form R, or any nonholonomic deformation to $\widehat{R}$ with respect to general N–adapted frames. Nevertheless, we can prescribe any

values of constants H_0 and H_1 and of ω and compute effective dark energy and dark matter oscillating cosmology effects for any off–diagonal solution in massive gravity and/or effective MGT, GR. To extract contributions of $\mathring{\mu}$ we can fix, for instance, $\widehat{f}(\widehat{R}) = \widehat{R} = R$ and using (1) and (2) we can relate $f(R(t))$ and respective constants to certain observable data in cosmology.

The MGT theories studied in this work encode, for respective nonholonomic constraints, the ekpyrotic scenario which can be modelled similarly to $f(R)$ gravity. A scalar field is introduced into usual ekpyrotic models in order to reproduce a cyclic universe and such a property exists if we consider off–diagonal solutions with massive gravity terms and/or f–modifications. The main idea is to develop the reconstruction techniques for the scalar–tensor theory using the AFDM, with nonholonomic off–diagonal metric and linear connection deformations. Working with general classes of off–diagonal cosmological solutions, the problem is to state the conditions for generating and integration functions when a corresponding ekpyrotic scenario will "survive" for certain constraints and in respective limits.

Let us consider a prime configuration with energy–density for pressureless matter $\mathring{\rho}_m$, for radiation and anisotropies we take respectively $\mathring{\rho}_r$ and $\mathring{\rho}_\sigma$ for radiation and anisotropies, κ is the spatial curvature of the universe and a target effective energy–density ρ (23). A FLRW model can be described by

$$3H^2 = 8\pi[\frac{\mathring{\rho}_m}{a^3} + \frac{\mathring{\rho}_r}{a^4} + \frac{\mathring{\rho}_\sigma}{a^6} - \frac{\kappa}{a^2} + \rho].$$

We generate an off–diagonal/massive gravity cosmological cyclic scenario containing a contracting phase by solving the initial problems if $w > 1$, see (24). A homogeneous and isotropic spatially flat universe is obtained when the scale factor tends to zero and the effective f–terms (massive gravity and off–diagonal contributions) dominate over the rests. In such cases, the results are similar to those in the inflationary scenario. For recovering (25), the ekpyrotic scenario takes place and mimic the observable universe for $t \sim \pi/2\omega$ in the effective EoS parameter

$$w \approx -1 + \sin\omega t/3\omega H_1 \cos^2\omega t \gg 1.$$

This allows us to conclude that in massive gravity and/or using off–diagonal interactions in GR cyclic universes can be reconstructed in such forms that the initial, flatness and/or horizon problems can be solved. We can compute possible locally anisotropic and inhomogeneous small contributions for self–consistent models with nonholonomic frames.

In the diversity of off–diagonal cosmological solutions which can constructed using above presented methods, there are cyclic ones with singularities of the type of big bang/ crunch behaviour. This is still a largely unexplored area both for the geometric methods of constructing exact solutions of PDEs and recovering procedures for certain "preferred" fundamental physical objects compatible with experimental data. Choosing necessary types generating and integration functions, we can avoid singularities and elaborate models with smooth transition. Using the possibility to generate nonholonomically constrained f–models with equivalence to certain classes of solutions in massive gravity and/or off–diagonal configurations in GR, we can study in this context, following methods in [3, 12], big and/or little rip cosmology models, when the phantom energy–density is modelled by off-diagonal interactions. We note that such nonsingular models for dark energy were proposed as alternatives to ΛCDM cosmologies, [20]. Using corresponding classes of generating functions, we can reproduce the sceanaros with a phantom scalar modeling a little rip in the framework of AFDM and nonholonomic MGTs. We omit such considerations in this article (see a summary of such construction and further developments in [15]).

6 Conclusions

In this paper we deal with new classes cosmological off–diagonal solutions in massive gravity with flat, open and closed spatial geometries. These solutions can be systematically constructed for various types of modified gravity theories, MGTs, and in general relativity, GR. We applied an advanced geometric techniques for decoupling the field equations and constructing exact solutions in massive and zero mass $f(R)$ gravity, theories with nontrivial torsion and noholonomic constraints to GR and possible extensions on (co) tangent Lorentz bundles. The so–called anholonomic frame deformation method, AFDM, was elaborated during last 15 years in our works [12, 17], where a number of examples of off–diagonal solutions and new applications in gravity and modern cosmology were considered.

A very important property of such generalized classes of cosmological solutions is that they depend, in general, on all spacetime coordinates via generating and integration functions and constants. They describe certain models of inhomogeneous and locally anisotropic cosmology with less clear physical meaning and possible physical implications [5]. After some classes of solutions were constructed in a most general form, we can impose at the end additional nonholonomic constraints, cosmological approximations, extract configurations with a prescribed spacetime symmetry and/or dependence on certain mass parameters, consider asymptotic conditions etc. Thus, our solutions can be used for elaborating homogeneous and isotropic cosmological models with arbitrary spatial curvature, to study generalized Killing and non-Killing, with possible nonholonomically deformed (super) symmetries [11, 16] and to study "non–spherical" collapse models of the formation of cosmic structure such as stars and galaxies (see also [14]).

Another aspect of the AFDM is that if we work only with cosmological scenarios for diagonalizable metrics, there are possibilities to discriminate the massive gravity theory form the f–gravity and/or GR. For diagonal metrics depending on a time like coordinate, we can formulate mathematical cosmology problems for certain nonlinear systems of ordinary differential equations with general solutions depending on integration constants. Identifying such a constant, for instance, with a graviton mass parameter, we do not have much possibilities to mimic a number of similar effects in GR for different MGTs. Following only such a "diagonal" approach, we positively have to modify the GR theory in order to explain observational data in modern cosmology and elaborate realistic quantum models of massive gravity.

We now discuss a new and important feature of the off–diagonal anisotropic configurations which allows us to model cosmic accelerations and massive gravity and/or dark energy and dark matter effects as certain effective Einstein spaces. Having integrated such system of nonlinear partial differential equations, PDEs, for a large class of such solutions, we can put and motivate such questions: May be we do not need to modify radically the GR theory but only to extend the constructions to off–diagonal solutions and nonholonomic systems and try to apply this in modern cosmology? Could we explain observational data in modern cosmology via nonlinear diagonal and/or off–diagonal interactions with non–minimal coupling for matter and/or different phases of massive and zero mass gravity. This is a quite complicated theoretical and experimental problem and the main goal of this and our recent papers cited in Refs. [12, 15] was to analyze such constructions from the viewpoint of massive gravity theory when off–diagonal effects can be alternatively explained to other types of gravity theories.

The reconstruction procedure for cosmological models with non–minimally coupled scalar fields evolving on a flat FLRW background and in different MGTs was studied in [4, 3]. In this work, we elaborated a reconstruction method for the massive gravity theory which admits an effective off–diagonal interpretation in GR and f–modified gravity with cyclic and ekpyrotic universe solution. We concluded that the expansion can be around the GR action even if we admit a nontrivial effective torsion. For zero torsion constraints, it is possible to construct off–diagonal cosmological

models keeping the approach in the framework of the GR theory. We further investigated how our results indicate that theories with massive gravitons, with possible f–modified terms and off–diagonal interactions may lead to more complicated scenarios of cyclic universes. Following such nonlinear (off–diagonal) approaches, the ekpyrotic (little rip) scenario can be realized with no need to introduce additional fields (or modifying gravity) but only in terms of massive gravity or GR. Another interesting constructions can be related to reconstruction scenarios of $f(R)$ and massive gravity theories leading to little rip universes both in locally anisotropic and isotropic variants. Finally, we note that the dark energy for little rip models present an example of non–singular phantom cosmology.

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References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211

[2] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D. 64 (2001) 123522; J. K. Erickson, D. H. Wesley, P. J. Steinhardt and N. Turok, Phys. Rev. D 69 (2004) 063514; E. I. Buchbinder, J. Khoury and B. A. Ovrut, Phys. Rev. D. 76 (2007) 123503; P. J. Steinhardt and N. Turok, Science 296 (2002) 1436

[3] K. Hinterbichler, Rev. Mod. Phys. 84 (2012) 671; S. Nojiri, S. Odintsov and N. Shirai, JCAP 1305 (2013) 20; S. Nojiri and S. Odintsov, Phys. Lett. B716 (2012) 377; S. Nojiri, S. Odintsov and D. Saez-Gomez, AIP Conf. Proc. 1458 (2011) 207, arXiv: 1108.0767; Yi-Fu Cai and E. N. Saridakis, Phys. Rev. D 90 (2014) 063528

[4] A. Yu. Kamenshchik, A. Tronconi, G. Ventruri and S. Yu. Vernov, Phys. Rev. D87 (2013) 063503; J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Phys. Lett. B732 (2014) 380

[5] A. P. Kouretsis, Eur. Phys. J. C74 (2014) 2079; S. Basilakos, A. P. Kourtesis, E. N. Saridakis and P. Stavrinos, Phys. Rev. D88 (2013) 123510; P. Stavrinos and S. Vacaru, Class. Quant. Grav. 30 (2013) 055012; E. Guendelman, H. Nishino and S. Rajpoot, Phys. Lett. B732 (2014) 156; E. I. Guendelman and D. Singleton, JHEP 1405 (2014) 096; E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, Phys. Lett. B718 (2013) 1099

[6] K. Hinterbinchler, Rev. Mod. Phys. 84 (2012) 671; C. de. Rham, Living Rev. Rel. 17 (2014) 7

[7] H. van Dam and J. J. G. Veltman, Nucl. Phys B 22 (1970) 397; V. Zakharov, JETP Letters (Sov. Phys.) 12 (1970) 312; A. I. Vainshtein, Phys. Lett. B 39 (1972) 393

[8] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 2008; C. Deffayet, G. R. Dvali and G. Babadaze, Phys. Rev. D 65 (2002) 044023; M. A. Luty, M. Porrati and R. Rattazzi, JHEP 0309 (2003) 029; A. Nicolis and R. Rattazzi, JHEP 0406 (2004) 059

[9] D. G. Boulware and S. Deser, Phys. Rev. D 6 (1972) 3368; N. Arkani–Hamed, H. Gergi and M. D. Schwarz, Annals Phys. 305 (2003) 96

[10] C. de Rham, Phys. Lett. B 681 (2009) 89; C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020; C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106** (2011) 231101; S. Hassan and R. A. Rosen, Phys. Rev. Lett. 108 (2012) 041101; JHEP 1204 (2012) 2012

[11] A. Salam and J. A. Stradhee, Phys. Rev. D 16 (1977) 2688; T. Gheorghiu, O. Vacaru and S. Vacaru, Eur. Phys. J. C74 (2014) 3152

[12] S. Vacaru, Eur. Phys. J. C 74 (2014) 3132; Int. J. Mod. Phys. D 21 (2012) 1250072; Gen. Relat. Grav. 44 (2012) 1015; Europhys. Letters (EPL) 96 (2011) 5001; Class. Quant. Grav. 28 (2011) 215991; Int. J. Geom. Meth. Mod. Phys. 8 (2011) 9; J. Math. Phys. 49 (2008) 043504; S. Vacaru and D. Singleton, Class. Quant. Grav. 19 (2002) 3583-3601; S. Vacaru and F. C. Popa, Class. Quant. Grav. 18 (2001) 4921-4938

[13] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, Phys. Rev. D 84 (2011) 124046; A. E. Gumrukcuoglu, C. Lin, and S. Mukoyama, JCAP 1203 (2012) 006; M. von Strauss, A. Schmidt–May, J. Enander, E. Mortsell and S. F. Hassan, JCAP 1203 (2012) 042; M. S. Volkov, JHEP 1201 (2012) 035; S. Koyama, G. Niz, and G. Tasinato, JHEP 1112 (2011) 065; P. Gratia, W. Hu and M. Wyman, Phys. Rev. D 86 (2012) 061504

[14] T. Kobayashi, S. Siino, M. Yamaguchi and D. Yoshida, Phys. Rev. D 86 (2012) 061505

[15] S. Vacaru, Off-Diagonal Ekpyrotic Scenarios and Equivalence of Modified, Massive and/or Einstein Gravity, arXiv: 1304.1080; E. Elizalde and S. Vacaru, Einstein Spaces Modeling Non-Minimal Modified Gravity, arXiv: 1310.6868

[16] R. Geroch, J. Math. Phys. 12 (1971) 918; R. Geroch, J. Math. Phys. 12 (1972) 394; S. Vacaru, Int. J. Geom. Meth. Mod. Phys. 4 (2008) 1285

[17] S. Vacaru, Acta Applicandae Mathematicae 110 (2010) 73; S. Vacaru and F. C. Popa, Class. Quant. Grav. 18 (2001) 4921; S. Vacaru and D. Singleton, Class. Quant. Grav. 19 (2002) 2793

[18] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gababadze and A. J. Tolley, Phys. Rev. D 85 (2012) 044024; K. Koyama, G. Niz and G. Tasinato, JHEP 1112 (2011) 065

[19] B. B. Kadomtsev and V. I. Petviashvili, Doklady Akademii Nauk SSS 192 (1970) 753; Sov. Phys. Dokl. 15 (1970) 539 [English translation]

[20] P. H. Frampton, K. J. Ludwick and R. J. Schrerrer, Phys. Rev. D84 (2011) 063003; P. H. Frampton, K. J. Ludwick, S. Nojiri, S. D. Odintsov and R. J. Schrerrer, Phys. Lett. B708 (2012) 204-2011; E. Elizalde and D. Saez-Gomez, Phys Rev. D. 80 (2009) 044030